Introduction to low dimensional topology

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October 27, 2015

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Preface

Low dimensional topology is the study of topology in dimensions 1,2,3 and 4. There is a dichotomy between these low dimensions and higher dimensions ≥ 5. This was first illustrated by Smale. The Poincare conjecture predicted that any closed simply connected manifold was homeomorphic to the sphere. In dimension 2 this easily follows from the classification of surfaces. For a long time people thought the next step would be dimension 3 until Smale proved the conjecture in dimensions ≥ 5.

Chapter 1

Knots, links, braids and tangles, oh my!

1.1 The 3-sphere

In the first chapter I'll be talking about knots and their various relatives. Knots naturally live in the 3-space but because it is not compact we will use the 3-sphere instead which is the unit sphere in R^4. You can think of it topologically without using R^4 as the 3-ball with its boundary identified to a point. There are other interpretations like if you take two balls and identify their boundaries (two copies of S^2) using an orientation reversing homeo. It is identity on latitude but sends longitude to its negative.

You can also take two solid tori and do the same thing. The last two examples are Heegaard decompositions of 3-sphere.

1.2 Knots and links

Definition 1.2.1. *A knot is a smooth embedding $\phi : S^1 \to S^3$ up to reparametrization. An oriented knot is the same thing when we've chosen an orientation on the domain.*

Up to reparametrization mean that if $u : S^1 \to S^1$ is a diffeomorphism (i.e. reparametrization) then we regards ϕ and $\phi \circ u$ to represent the same knot. In other words a knot is a closed connected smooth submanifold of S^3.

Examples: unknot, trefoil (DNA trefoil), figure eight

We require smoothness to avoid *wild knots* with fractal behavior as in figure 1.2.2.

Exercise 1.2.2. *Show that there are no wild knots in two dimensions.*

Definition 1.2.3. *A link with k components is a smooth embedding of a disjoint union of k copies of S^1 to S^3. An orientation on a link is given by choosing*

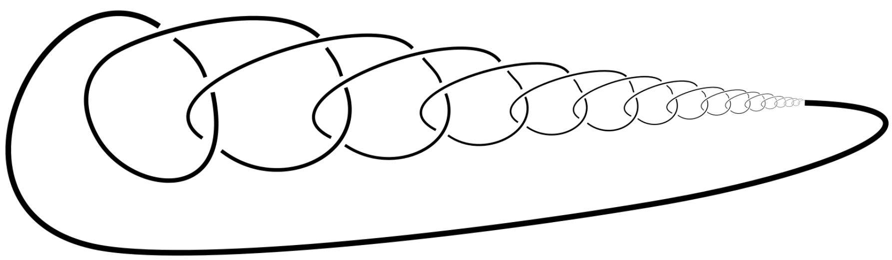


Figure 1.1: A wild knot

orientations on each S^1 component in the domain.

A link with k components admits 2^k different orientations. Of course a link with a single component is a knot.

Examples: Hopf link, Borromean link

We often consider a link as the image of the embedding.

Example 1.2.4. *Torus knots and links: $T(p, q)$ is the intersection of $x^p = y^q$ in $\mathbb{C}^2$ with the unit sphere. It can be viewed as wrapping around a torus p times horizontally and q times vertically. If $(p, q) = 1$ this is a knot and otherwise a link. Knots that can be obtained this way are called algebraic.*

Exercise 1.2.5. *Consider the Hopf fibration $p : S^3 \to S^2$ given by $p(z_1, z_2) = (2z_1z_2^*, |z_1|^2 - |z_2|^2)$. Show that for any two points $a, b \in S^2$ the union $p^{-1}(a) \cup p^{-1}(b) \subset S^3$ is a Hopf link.*

Any two links with the same number of components are homeomorphic to each other. So we need a different notion of equivalence of links.

Definition 1.2.6. *Two links K, L are equivalent if there is an orientation preserving homeomorphism ϕ of S^3 such that $\phi(K) = L$. If in addition K, L are oriented then we require $\phi(K)$ to have the same orientation as L.*

Def of orientation preserving for contin and smooth ϕ. Why we differentiate between non-orient preserv?

Intuitively it seems that trefoil is not isotopic to unknot but the actual proof is not easy. We will need knot invariants.

The complement of a knot is S^3 minus a tubular neighborhood of the knot. So it's a 3-manifold with a torus boundary component.

An equivalence class is called a *knot type*. Knot types can be seen as open cells in the space of all immersions $S^1 \to S^3$. The walls between cells are *singular knots*.

Definition 1.2.7. *Two links K, L are isotopic if there is a one parameter family ϕ_t of orientation preserving homeomorphisms ϕ_t of S^3 such that $\phi_0 = id$ and $\phi_1(K) = L$.*

In Intuitively an isotopy is a continuous deformation of the know in such a way that at all times t the knot $\phi_t(K)$ doesn't intersects itself.

Ex: A knot is isot to unknot if it bounds an embedded disk in S^3.

Proposition 1.2.8. *Two links K, L are equivalent iff they are isotopic.*

Proof. If they are isotopic then by definition there is a one parameter family ϕ_t so that $\phi(K) = L$ so K, L are equivalent. For the reverse we need a theorem of Fisher which says if ϕ is an orientation preserving homeomorphism of a closed oriented 3-manifold then ϕ is isotopic to identity. □

Isotopies are isomorphisms between links i.e. they don't change link type. You may ask what about general morphisms between links, morphisms between different link types. Those are given by *link cobordisms* whose study we postpone until

Further Remarks

One can consider knots and links in higher dimensions as well. As an easy exercise you can argue that any link embedded in $\mathbb{R}^4$ is isotopic to the unlink. However if we look at embeddings of S^2 to S^4 then we are in for nontrivial knot types.

1.3 Link projections and diagrams

Definition 1.3.1. *A polygonal link is a link which consists of finitely many straight line segments in $\mathbb{R}^3$.*

Theorem 1.3.2. *Every (polygon) link L is equivalent to a polygon link.*

For each $p \in L$ let N_pL be the normal plane to L at p and $B_\epsilon(p) \subset N_pL$ be a disk of Euclidean radius $\epsilon > 0$. $B_\epsilon L = \cup_{p\in L} B_\epsilon(p)$ is called a tubular neighborhood of L.

Proof. □

A projection is a linear surjective map $\mathbb{R}^3 \to \mathbb{R}^2$. Such a projection is determined by a line in $\mathbb{R}^3$ so the set of all such projections can be identified by $\mathbb{R}P^3$.

Definition 1.3.3. *For a link $L \subset \mathbb{R}^3$ a projection ϕ is regular if $\phi|_L$ is an embedding outside a finite number of double points. If L is polygonal then we require ϕ to be 1-1 on the set of vertices of L.*

Theorem 1.3.4. *Every piecewise linear link L admits infinitely many regular projections to the plane. Moreover the set of the regular projections for L, is dense in $\mathbb{R}P^2$.*

In other words any projection map can be perturbed slightly to become regular for L.

The image of such a regular projection is called a link diagram for L.

Definition 1.3.5. *A link diagram for L is an immersion $\psi \to \mathbb{R}^2$ which is an embedding outside a finite number of double points (called crossings). Together with a specification of an over or under crossing at each crossing.*

Any link diagram gives us an isotopy class of links. This is given by viewing the diagram as lying in $R^2 \times \mathbb{R}$ and raising or lowering the diagram at each crossing according to whether

Conversely given a link L we can isotope it to a polygonal link using using theorem **??** and then using theorem **??** to obtain link diagrams for it. So we can represent knots by knot diagrams and we know when two diagrams represent the same knot type.

Now the question we are faced with is how equivalence of links translate into their diagrams. The answer is given by a theorem of Reidemeister.

Theorem 1.3.6. *Any two projections of two isotopic polygon links L_0, L_1 are related to each other by a sequence of Reidemeister moves together with plane isotopies.*

Theorem 1.3.7 (Moise)**.** *Two polygon links are equivalent iff they are related by a sequence of piecewise linear maps.*

Idea: 1-For each polygonal link L there is a triangulation of S^3 such that L is a union of edges of triangles in this triangulation. 2-For any triangulation Δ of S^3 and any homeomorphism $\phi : S^3 \to S^3$ we can refine Δ to a Δ' and isotope *phi* to ϕ' such that ϕ' is simplicial w.r.t. Δ'.

Proof. It is clear that a Reidemeister move preserves the link type.

□

Definition 1.3.8. *The crossing number of a link is the minimum number of double points in a diagram for the link.*

1.4 Operations on knots

1.4.1 Reverse

The reverse of an oriented knot is given by reversing its orientation.

A knot is called invertible if it is equivalent to its reverse. Trefoil and figure eight knots are invertible. (You can rotate them.) The first example of a non-invertible knot was discovered by Trotter. His proof uses hyperbolic geometry.

1.4.2 Mirror of a link

The reflection or the mirror of a knot is given by applying an orientation reversing diffeo of S^3 to it. (The choice doesn't matter because the group of all such diffeos is connected.) We can choose the diffeo which sends (x, y, z) to $(x, y, -z)$ to it changes all over crossings to under crossings and vice versa.

A link is called chiral if it is not equivalent to its mirror and amphichiral otherwise. Figure eight is amphichiral (using explicit deformation) but trefoil is not (using Jones polynomial).

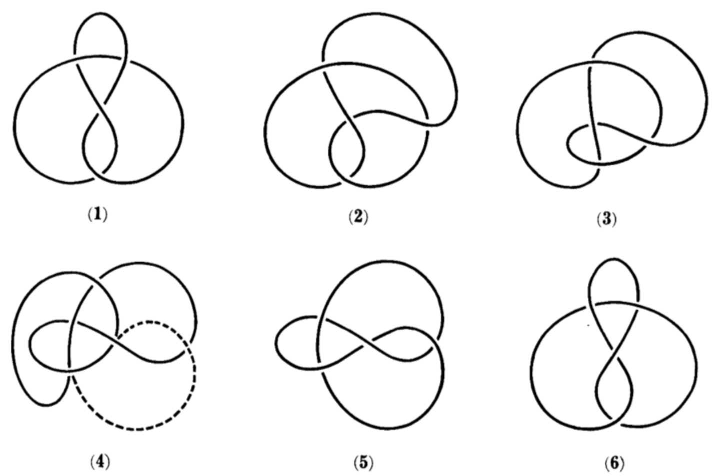


Figure 1.2: Isotoping the figure eight knot to its mirror

1.4.3 Connected sum of two knots

The connected sum of two knots can be defined easily. It is well defined and abelian.

Definition 1.4.1. *A knot is prime if it is not the unknot and if it is isotopic to a to a direct sum $K_1 \# K_2$ then one of the K_i is the unknot.*

Example: Trefoil is prime because the only nontrivial decomposition for it is into sum of a knot with 1 crossing and one with two and there are no such knots.

Theorem 1.4.2. *Any knot can be expressed uniquely as a direct sum of a finite number of prime knots.*

When a knot K is not prime then it can be decomposes as $K_1 + K_2$. If any of the K_1, K_2 is not prime then they can be written as sums of other knots and so on. The question is why should this process end in finitely many steps. You may think the crossing numbers of K_i is strictly less than those of K but this has not been proved.

Conjecture 1.4.3.

$$c(K+L) = c(K) + c(L). \tag{1.1}$$

Theorem 1.4.4 (Marc Lackenby)**.**

$$c(K+L) \geq \frac{c(K)+c(L)}{152}. \tag{1.2}$$

Instead in the next section we will define a new invariant of knots called genus which is additive w.r.t. sums and we use it to show prove the decomposition theorem.

1.4.4 Satellite operation

Satellite operation is a far reaching generalization of connected sum.

1.5 Seifert surfaces and the proof of the decomposition theorem

l only prove existence and not uniqueness. To prove it we assign a numerical invariant to knots called genus and show that it is additive w.r.t. direct sum. Genus and Seifert surfaces are of central importance to knot theory.

Theorem 1.5.1 (Seifert)**.** *For any oriented link K there is a compact connected orientable surface (called a Seifert surface) F s.t. $\partial F = K$ with the induced orientation.*

Definition 1.5.2. *A resolution of a crossing*

Seifert Algorithm. Start with a diagram D for L. Resolve all the crossings in an orientation preserving way. We are left with a number of simple closed curves in the plane. Imagine each one of these closed curves bounds a disk. Then in place of each erased crossing attach a twisted ribbon connecting the corresponding disks. This way we obtain a surface S. It is orientable because it has two sides. □

Note that there can be Seifert surfaces for a link which cannot be obtained from applying the Seifert algorithm to any diagram for the link.

Example 1.5.3. *For unknot (genus 0,1), trefoil, Hopf link For links the outcome of the algorithm depends on the orientation.*

Starting with any Seifert surface S for L you can attach handles to it to obtain a higher genus Seifert surface for the same link. On the other hand we can not arbitrarily decrease the genus of a Seifert surface for a fixed link.

Definition 1.5.4. *The minimum genus of a Seifert surface for K is called the genus of K.*

Note that it is possible that this lowest genus surface may not be obtained from Seifert algorithm applied to any projection of the link.

For example $g(unknot) = 0$ and any link with genus zero is equivalent to the unlink.

For the Seifert surface F obtained from the Seifert algorithm the Euler characteristic is $v - e + f$ and in terms of genus it equals $2 - \#\partial F - 2g(F)$. If d, b are the number of disks and bands respectively then $v = 4b, e = 6b, f = b + d$ so $\chi(F) = d - b$. Therefore $2g(F) = 2 - d + b - \#K$. So the Seifert surface we got for trefoil has genus one.

From this we see that twist knots all have genus one (assuming they are not equivalent to the unknot). We can also deduce that $g(K) \leq c(K)/2$.

Seifert surface for trefoil using Seifert algorithm

Theorem 1.5.5. *For any two knots K, L we have $g(K + L) = g(K) + g(L)$.*

Proof. If we have two Seifert surfaces for K and L (which are two surfaces with a single boundary component) we can take their connected sum to get a Seifert surface for $K\#L$. This gives $g(K\#L) \leq g(L) + g(L)$.

Now for the reverse inequality pick a Seifert surface F for the sum with minimal genus. Consider a sphere S containing K and intersecting the sum in only two points. We can assume S intersects F transversely and so in a finite number of intervals and a single arc β which connects the two endpoints. If $F \cap S = \beta$ then F is a connected sum of a Seifert for K and one for L so $g(F) = g(K) + g(L)$. We show that we can surger out pieces of F to turn the intersection into a mere β.

Now the circles in the intersection are disjoint so we can take the innermost one. By Jordan curve theorem it decomposes S^2 into two discs. We cut F along the inner disk and attach two parallel disks instead. This way we get a new surface F_1 which intersects S in one less component. F_1 is compact and orientable and so, if connected, it is a Seifert surface for the sum. But connectivity implies $g(F_1) < g(F)$ contradiction. So F_1 is disconnected and we can take its connected component which contains $K\#L$. We can repeat this argument to remove all circle components and get the connected sum Seifert Surface with genus $g(K) + g(L)$. □

Corollary 1.5.6. *If $K\#L \simeq \bigcirc$ then both K, L are unknots. So no nontrivial knot has an inverse w.r.t. $\#$.*

If we have a knot K which is not prime then $K \simeq K_1\#K_2$. If one of the K_i is not prime then it decomposes and we can repeat this process. We can use genus to argue that this process won't go forever and will stop after a finite number of iterations.

1.6 Fundamental problems of knot theory

Equivalence problem

Given two knot diagrams are they equivalent?

Unknot detection

In particular is a given knot diagram equi to the unknot? For example is trefoil equi to unknot? Figure 1.6 shows a complicated diagram for the unknot.

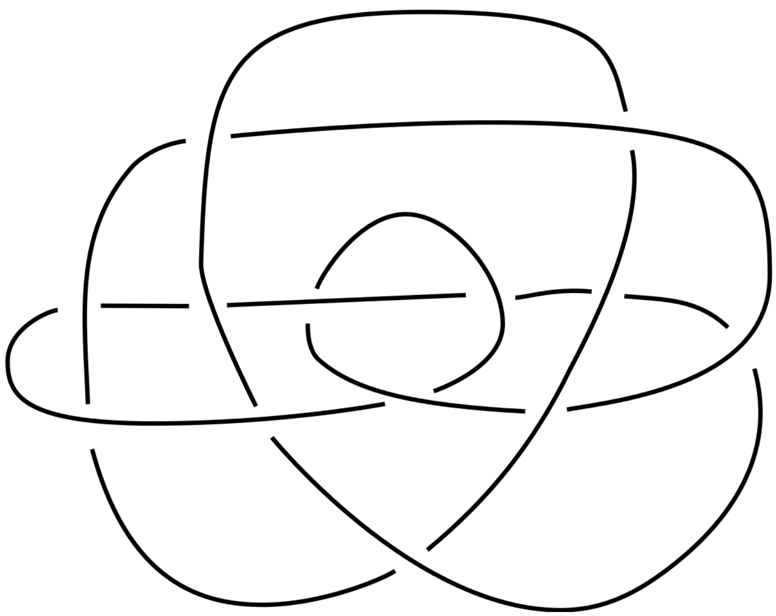


Figure 1.3: Ochiai unknot

Tabulating knots and links

Rolfsen table

Unknotting number

1.7 Braid groups and links

In this section we want to introduce a family of groups $\{Br_m\}_{m\in\mathbb{N}}$ called braid groups and an equivalence relation $\sim$ on them called Markov relation such that $\bigcup_m Br_m/\sim$ is the set of all link types.

Definition 1.7.1. *A braid on m strands is an embedding of m line segments in $\mathbb{C} \times [0,1]$ in such a way that the strands don't have horizontal tangents and the endpoints are mapped to $\{1,\ldots,m\} \times \{0,1\}$. We consider braids up to isotopy.*

Braids can be composed by concatenation. We have an identity braid and the inverse of a braid is given by flipping it vertically. Associativity is obvious. So we get a group Br_m.

Br_m has a presentation due to Artin.

Theorem 1.7.2. *Generators are $\sigma_1, ..., \sigma_{m-1}$ and the relations are A1) $\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i$ if $|i-j| > 1$ and A2)$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ in addition to A3) $\sigma_i \sigma_i^{-1} = 1$.*

Proof. Idea of proof. Seems to be exactly the same as proof of Reidemeister Define a group G with generators $x_1, \ldots, x_{m-1}$ and relations . Define a homomorhism $\phi : G \to br_m$ given by $\phi(x_i) = \sigma_i$. ϕ is obviously onto so we need to show that it is 1-1. Let g be an element of G such that $\phi(g) = 1$ and $g = x_{i_1}^{k_1} \cdots x_{i_n}^{k_n}$ be a word describing it. $\phi(g) = \sigma_{i_1}^{k_1} \cdots \sigma_{i_n}^{k_n}$ is a braid, given as a composition of elementary braids which is isotopic to identity. So there is a sequence of Δ moves that turns $\phi(g)$ into the identity braid.

So the proof is reduced to showing that if two braid words w_1, w_2 are related by a Δ move then one can be transformed to the other by a sequence of relations A1- A3.

As in the proof or Reidemeister's theorem we must look at different cases for what the triangle ABC which turns into the line AB in the Δ move contains.
Case 1: It's empty.
Case 2: It contains one line entering and exiting ABC. This corresponds to A1 or A3.
Case 3: It contains a crossing. This corresponds to A2.
Case 4: It contains many lines and or crossings. Then we can subdivide ABC into small triangles such that each one contains only one line or crossing. □

Corollary 1.7.3. *The group Br_2 is isomorphic to $\mathbb{Z}$.*

Corollary 1.7.4. *Each one of Artin generators $\sigma_i \in Br_m$ has infinite order.*

Proof. Imagine $\sigma_i^k = e$ for some k. Then $\sigma_i^k = 1$ is isotopic to identity. In particular the i'th and $(i+1)$'st strands of this braid alone can be isotoped to the identity braid on two strands. But those two strands form a $\sigma_1 \in Br_2$ and so σ_1 would be isotopic to identity which is contradiction. □

Corollary 1.7.5. *The homomorphism $Br_m \to Br_n$ for $m \leq n$ which sends the ith generator in the domain to the ith generator to the target is 1-1.*

Proof is similar to the proof of the above corollary.

If you quotient Br_m with $\sigma_i^2 = e$ for all i then you'll get the symmetric group. In a poetic language elements of the symmetric group are intrinsic permutations while elements of the braid group are extrinsic ones.

Definition 1.7.6. *The pure braid $P_m < Br_m$ consists of braids which do not permute the elements.*

The pure braid group P_m is iso to the π_1 of $\mathbb{C}^m \backslash \{z_i = z_j\}$ and Br_m is isomorphic to the π_1 of the quotient of this space by the action of the symmetric group.. The latter space is called configuration space.

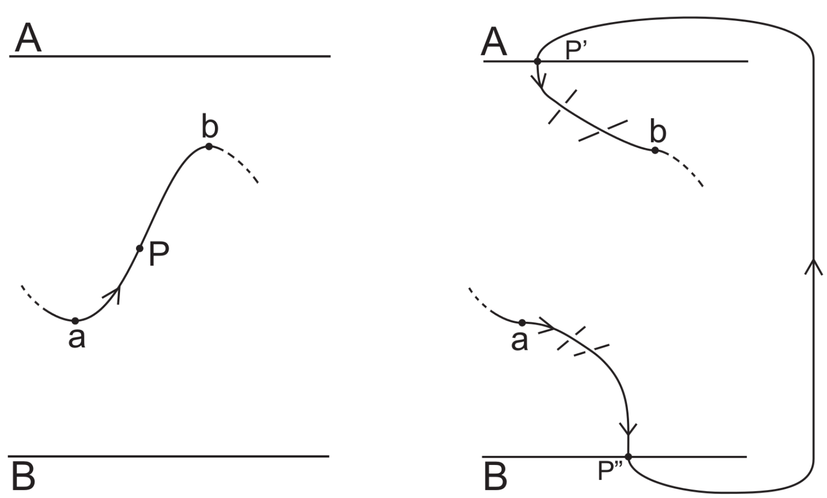
We can identify the top and bottom endpoints of a braid to get a link. This is called link closure.

Theorem 1.7.7. *(Alexander) Any link K is isotopic to the closure of a braid β.*

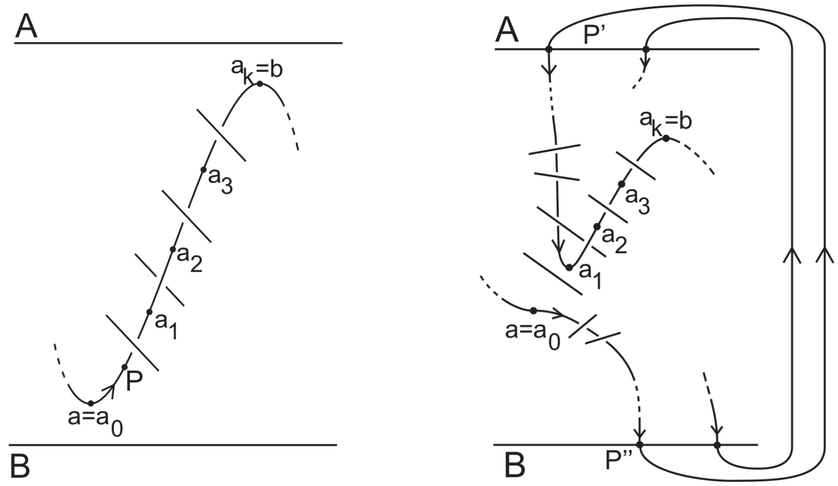
Proof involves getting rid of the maxima and minima of the y-coordinate on the link diagram.

Proof. Start with a diagram D for K. The y-coordinate restricted to D has some minima and maxima. We draw a square S containing D and will try to isotope D to move all these extrema outside of the square. Then the y coordinate has no extrema inside S and so what lies inside S would be a braid. Let a, b be minimum and maximum respectively for y such that there is no other extrema in between them when we move along K from a to b in the direction in which y increases. Let $a_0 = a, a_1, \ldots, a_k = b$ be points on this strand such that between a_i, a_{i+1} there is exactly one intersection of K. There are two cases.

Case I: $k = 1$. We cut K at a point in between a, b and move the endpoints to the S, in such a way that they pass over all other strands. We connect the endpoints by an arc outside S.



Case II: $K > 1$. We start by doing a similar procedure like in Case I for $a = a_0$ and a_1. I.e. we cut K at a point P between a_0, a_1 (which is not the crossing) and then move the endpoints to the upper and lower edges of S. This time we should note whether the strand that we cut goes over or under at the crossing between a_0, a_1. If goes above, the segments a_0, P and P, a_1 go above all other strands and vice versa. Then we move to a_1, a_2 and after repeating this procedure a finite number of times we'll be in Case II.



We do this procedure for all strands of K connecting a min to a max and after that the diagram we get is the closure of a braid inside S.

The reason this diagram is isotopic to the original one is that we can move the segments outside S back inside S one by one to get the original diagram. □

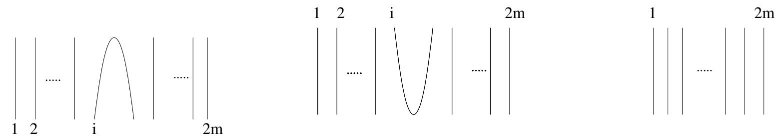
Example 1.7.8. *The torus knot $T(p, q)$ is given by the closure of the braid $(\sigma_1\sigma_2\cdots\sigma_{p-1})^q$.*

Theorem 1.7.9. *(Markov) Two braids have isotopic closures iff they are related by one of the two Markov move: Conjugation or (de)stabilization (adding one extra strand with a twist).*

It is easy to see that Markov moves do not change the type of the braid closure. Markov one is equivalent to RII, Markov 2 is equi to RI and RIII is equi to the braid relation. The proof of the reverse direction is much more difficult because after applying a Reidemeister move you may no longer have a braid closure.

Exercise 1.7.10. *Let L be the closure of $\beta \in Br_m$ and L' be the 180 degree rotation of L which is the closure of a braid β'. Find a sequence of Markov moves that sends β to β'. Do the same for turning L upside down.*

Exercise 1.7.11 (Burau representation)**.** *Consider the matrix $A(t) = \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}$ and let $\rho_i(t) = 1 \oplus \cdots \oplus 1 \oplus A(t) \oplus 1 \oplus \cdots \oplus 1$ where the number of 1's before $A(t)$ is $i-1$. Show that the map $\sigma_i \to \rho_i$ for $1 \le i \le m-1$ gives a representation of the braid group Br_m.*



1.8 Tangles

Tangles are what you get if you chop a link into pieces, or in other words a link with endpoints. They also generalize braids.

Definition 1.8.1. *An (m, n) tangle T is a 1 dimensional submanifold of $\mathbb{C} \times [0, 1]$ such that ∂T consists of $m+n$ points m of which are $\{1, \ldots, m\} \times \{0\}$ and n of them are $\{1, \ldots, n\} \times \{1\}$.*

An (l, m) tangle can be concatenated with an (m, n) tangle to give an (l, n) tangle. This way tangles form a category: objects are natural numbers and morphisms between m, n are (m, n)-tangles.

The elementary tangles are identity, caps, cups and elementary braids as in Figure.

Proposition 1.8.2. *Any tangle is isotopic to a composition of elementary tangles.*

Proof. Consider the restriction of the z coordinate to the tangle. The extrema of this function are given by caps, cups and the endpoints of the tangle. When we cut out the caps and cups we are left with a generalized braid (with extra endpoints) which is a composition of twists. ☐

Proposition 1.8.3 (Yetter [1])**.** *The following are all the commutation relations between elementary tangles. If $|i - j| > 1$ we have:*

$$\sigma_i \sigma_j = \sigma_j \sigma_i \tag{1.3}$$

$$\cap_i \cup_j = \cup_{j-2} \cap_i \tag{1.4}$$

$$\cap_i \sigma_j = \sigma_j \cap_i \qquad \cup_i \sigma_j = \sigma_j \cup_i, \tag{1.5}$$

and for any i we have:

$$\sigma_i \cup_i = \cup_i \tag{1.6}$$

$$\sigma_i \sigma_i^t = id \tag{1.7}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{1.8}$$

$$\cap_{i;m} \cup_{i+1;m} = id_{m-1} \tag{1.9}$$

$$\sigma_i \cup_{i+1} = \sigma_{i+1}^t \cup_i \qquad \sigma_i^t \cup_{i+1} = \sigma_{i+1} \cup_i \, . \tag{1.10}$$

1.8.1 Conway tangles

1.9 Families of knots

1.9.1 Torus knots

1.9.2 Alternating knots

1.9.3 2-bridge knots

1.9.4 Pretzel knots

1.9.5 Knots by their complements

1.9.6 Montesinos links

Chapter 2

Invariants of links

2.1 Invariants in general

By a link invariant we mean a function which assigns a unique value to any link type. This is equivalent to an assignment of a value to each link diagram in such a way that if two diagrams are related by a Reidemeister move then they get the same value. Similarly it can be given by assigning a value to each braid in such a way that is invariant under Markov moves. What we mean by "value" and "assignment" can be made precise using functors and categories.

Links invariants are useful for distinguishing between two different knot types.

We've already seen examples of knot and link invariants e.g. crossing number, genus, number of components, unknotting number for knots, unlinking number for links.

Evolution of link invariants: classically: numbers, 1900s: polynomials, 2000s: chain complexes.

2.2 Bridge number

A link diagram D is said to have bridge number k if it can be partitioned to a set of arcs $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k$ such that in all the crossings in D the α curves go over the β curves (and no crossings between α curves alone or β curves alone). We can think of the α curves as lying above a given plane P in space (and have their endpoints on P) while β curves lie below it.

Definition 2.2.1. *The bridge number of a link is the minimum of the bridge numbers of its diagrams.*

A 1-bridge knot is the unknot. Trefoil and figure eight are 2-bridge.

There are infinitely many 2-bridge knots. They can be described as follows. Let the curves α_1, α_2 be two line segments on the x-axis with end points A, B, C, D (from left to right). The curves β_1, β_2 join these endpoints without intersecting each other. We assume β_1 connects A to C and β_2 connects B to D. Let p be the number of segments to which each α curve is divided by β curves. (So β curves pass under each α_i $p-1$ times.) Let q be given as follows: start from B and follow β_2 untill you reach α_2 at a point Q and there are q other times β passes α_2 to the left of Q (including the one passing through C). Let us denote this knot (or link) by $\mathfrak{b}(p,q)$.

Theorem 2.2.2. *$\mathfrak{b}(p,q)$ and $\mathfrak{b}(p',q')$ are equivalent iff $p'=p$ and $q'=q^{\pm 1}$. Here q^{-1} is an integer between 1 and $2p-1$ such that $qq^{-1}=1 \mod 2p$.*

Theorem 2.2.3. *$\mathfrak{b}(p,q)$ is equivalent to the closure of the rational tangle given by $(a_1,\ldots,a_n)$ if p/q equals the continued fraction from $(a_1,\ldots,a_n)$. In particular two rational links are equivalent iff they have the same fraction.*

Theorem 2.2.4 (Schubert)**.**

$$br(K\#L) = br(K) + br(L) - 1 \tag{2.1}$$

A corollary of this theorem is that all 2-bridge knots are prime.

2.3 Linking number of two-component links

Imagine we have an oriented link L with two components J,K. Pick a diagram for this link. To each crossing between I,J we assign $+1$ if it is a right handed crossing (i.e. the strand heading right goes over) and -1 otherwise. Half the sum of these signs is called the crossing number of L and is denoted $lk(I,J)$. Note that lk does not take into account the crossings of individual I,J with themselves so it doesn't change if we flip such crossings.

Example for Hopf

Lemma 2.3.1. *Linking number is always an integer and we have $lk(I,J) = lk(J,I)$.*

Proof. Let n_1,n_2 be the number of positive crossings in which I resp. J goes over and n_3,n_4 are similarly the number of negative crossings. Then (if either I or J are simple closed curves) it follows from Jordan curve theorem that $n_1+n_3=n_2+n_4$ so $lk(I,J)=n_1-n_4=n_2-n_3=lk(J,I)$. □

Lemma 2.3.2. *Linking number is invariant under link isotopy.*

Proof. To show that linking number is invariant under isotopies of the link $J\cap K$ we look how it changes under Reidemeister moves. RI affects only one component so it doesn't change lk. For RII it is easy to see that the signs of the two crossings which are canceled are opposites. For RIII we have to consider different cases with different orientations on the strands and I leave that for you as an exercise. □

Example: Hopf link has linking number one. So it's not equivalent to unlink.
Image of whitehead link

Exercise 2.3.3. *Compute lk of Whitehead link in Figure* **??** *to be zero.*

So a two component link with linking number zero may be linked.

If we reverse the orientation of one of the components of L then its linking number is multiplied by -1. So if we reverse both components the linking number doesn't change.

Gauss: The linking number of two links can be given by an integral. If $r_1, r_2 : S^1 \to \mathbb{R}^3$ represent our two links then

$$lk(I,J) = \frac{1}{4\pi} \int_{S^1} \int_{S^1} \frac{r_1 - r_2}{||r_1 - r_2||^2} dr_1 \times dr_2 \tag{2.2}$$

We can use this theorem to prove the invariance of lk without using Reidemeister's theorem.

2.4 The Alexander polynomial from Seifert matrices

There are several different ways to define the Alexander polynomial of a link. In this section we look at one of them which uses Seifert matrices and in the following sections we will see two more.

Start with a Seifert surface F for a link L which has genus g. The first homology of F has $2g + r - 1$ generators. Pick a set of generators *and* a set of simple closed curves in F representing these generators. If a, b are two such curves on F we can push a a bit "down" F to get a closed curve a^- in S^3. To be more precise we can thicken the embedding of F in S^3 to an embedding $\phi : F \times [0,1] \to S^3$. This is because F is orientable and so has a tubular neighborhood . The thickening is chosen so that $\phi(F \times \{-1\})$ lies "below" F. This means that for any point $p \in F$ the vector from p to $\phi(p, +1)$ is in the direction of the orientation normal vector field to F. Then we take $a^- = \phi(a \times \{-1\})$.

Definition 2.4.1. *The Seifert matrix V of (K, F) has entries $lk(a, b^-)$ for a, b representing generators of $H_1(F)$.*

Definition 2.4.2. *The (Conway normalized) Alexander polynomial of K is given by*

$$\nabla_K(z) = \det(-tV + t^{-1}V^t). \tag{2.3}$$

where $z = t - t^{-1}$.

The reason the right hand side can be written as a polynomial in z is that if we call it $f(t)$ then $f(t^{-1}) = \det(-t^{-1}V + tV^t) = \det(-t^{-1}V^t + tV) = f(-t)$.

Note that we still don't know if this is a knot invariant. The above definition depends on a choice of a Seifert surface and a set of generators for it.

Example 2.4.3.

- *We obviously have $\nabla(\bigcirc) = 1$.*

- *For the Hopf link the Seifert algorithm yields a surface of genus zero whose H_1 has $0+2-1=1$ generators. If a denotes this generator then $lk(a,a^-)=-1$ so the Alexander polynomial is $\nabla_{Hopf}=z$.*

- *For the trefoil with its minimal genus Seifert surface F, $H_1(F)$ has two generators which we call a,b. We have $lk(a,a)=lk(b,b)=1$, $lk(a,b)=-1$, $lk(b,a)=0$. So Alex of trefoil is $det(-tV+t^{-1}V^t)=(t-t^1)^2+1=1+z^2$.*

Invariance of the Alexander polynomial

To prove invariance we must see how the Seifert matrix of a link L changes under
I) Changing the basis for the first homology of the Seifert surface (including changing the orientations of the generators).
II) Changing the Seifert surface.

When we do a change of basis, the Seifert matrix A changes to PAP^t where P is the change of basis matrix. It is an invertible matrix which has integer entries so $\det P \det P^{-1}=1$ so $\det P=\pm 1$. This clearly doesn't change the Alexander polynomial. The case of changing the Seifert matrix is more difficult and is the subject of the following definition and proposition. Recall that any two (abstract) oriented surfaces with the same number of boundary components can be converted to each other by a finite number of surgeries i.e. removing two disjoint disks from the surface and connecting the resulting two circles by a cylinder (or the reverse of this operation). Similar surgery operations can be done on embedded surfaces provided that the cylinder lies on one side of the surface.

Proposition 2.4.4. *If F,F' are two Seifert surfaces for a link L then there is a sequence $F_0=F,F_1,\dots,F_n=F'$ of Seifert surfaces for L such that for each i either F_i is obtained from F_{i-1} or F_{i-1} is obtained from F_i by an embedded surgery, or is obtained by an isotopy.*

Definition 2.4.5. *Let A be a square matrix and B one of the following:*

$$\begin{pmatrix} A & \xi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad or \quad \begin{pmatrix} A & 0 & 0 \\ \eta & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.4}$$

where ξ is an arbitrary column and η is a row. We call B an elementary enlargement of A and A an elementary reduction of B. Two square matrices are said to be S-equivalent if they are related by a sequence of elementary reductions or enlargements OR conjugation by a matrix P of determinant ± 1.

Lemma 2.4.6. *Any two Seifert matrices obtained from any two Seifert surfaces F,F' for a link L are S-equivalent.*

Proof. Let F' be obtained by an embedded surgery on F. Then the H_1 of F' has to more generators a, b than that of F. b can be taken to be a meridian of the cylinder that we attached to F to obtain F' and a can be taken to be its longitude. b is away from any generator f for $H_1(F)$ and so $lk(b, f^-) = lk(f, b^-) = 0$. It is easy to see that either $a \cup b^-$ is an unlink and $a^- \cup b$ is the Hopf link or the other way around. So $lk(a, b^-) = 0, lk(b, a^-) = 1$ or $lk(a, b^-) = 1, lk(b, a^-) = 0$. Why not -1? In the first case the Seifert matrix for F' is of the form

$$\begin{pmatrix} A & * & 0 \\ \eta & * & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.5}$$

which can be turned into a matrix of the form (2.4) by elementary row operations. The second case is similar. □

We leave it as an exercise for the reader to show that the value of $\det(A - tA^t)$ doesn't change when A is replaced by a matrix S-equivalent to it. From this we get that the Alexander polynomial is indeed a link invariant!

Corollary 2.4.7. *The trefoil is not isotopic to the unknot.*

Corollary 2.4.8. *For a knot K, $\deg \nabla_K \le 2g(K) \le c(K)$.*

Proposition 2.4.9.

$$\nabla_{K\#K'} = \nabla_K \cdot \nabla_{K'} \tag{2.6}$$

Proof. Pick Seifert surfaces S, S' for K and K'. The boundary connected sum $S\#S'$ gives a Seifert surface for $K\#K'$. If we pick bases for H_1 of S and S' their union gives a basis for $H_1(S\#S')$. This way the Seifert matrix of $K\#K'$ is the direct sum of those of K and k' and the result follows. □

Proposition 2.4.10. *The Alexander polynomial of a link with an unlinked component is zero. In particular the Alexander polynomial of the unlink is zero.*

Proof. We can have two disjoint connected surfaces F_1, F_2 bounding the two unlinked components. Let F be the connected surface obtained by making a tunnel between the two surfaces. If we take a basis for the H_1 of $F_1 \cup F_2$, we can obtain a basis B for $H_1(F)$ by adding a generator corresponding to the meridian e of the tunnel. We have $lk(e, f^-) = lk(f, e^-) = 0$ for any $f \in B$ including e itself. So the last row and column of the Seifert matrix are zero and so is the Alexander polynomial. □

Proposition 2.4.11. *The Alexander polynomial of a knot K is the same as those of its mirror K^* and its reverse $-K$.*

Proof. We show that $M_{-K} = M_K^t$ and $M_{K^*} = -M_K$. If we reverse the orientation of K then the orientation of its Seifert surface is reversed as well. Now the b^- of the new regime is b^+ of the old. So the (a, b) element in M_{-K} is $lk(a, b^+) = lk(a^-, b) = lk(b, a^-)$.

For the K^* we construct the Seifert surface using the Seifert algorithm. Each ribbon in this surface are twisted in the opposite way of the corresponding ribbon for the Seifert surface. This amounts to all crossings of a, b^- be flipped for any two homology generators a, b. This results in their linking number be multiplied by -1. (Use one of the ways of defining lk for each case.) □

Further remarks

- Alexander originally defined his polynomial by looking at the infinite cyclic cover of the knot complement. As we will see the H_1 of any knot complement $X = S^3 \backslash K$ is isomorphic to $\mathbb{Z}$. Therefore it has an infinite cyclic cover $\widetilde{X}$. The group of deck transformations of this covering is of course isomorphic to $\mathbb{Z}$ and let t be a generator for it. This way $H_1(\widetilde{X})$ can be regarded as a $\mathbb{Z}[t, t^{-1}]$-module called the Alexander ideal.

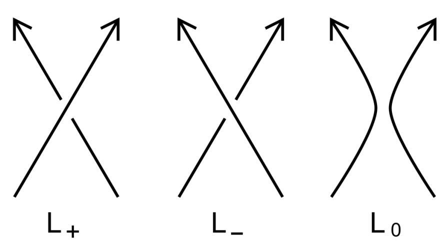
  The polynomial defined this way lives in $\mathbb{Z}[t^{1/2}, t^{-1/2}]$ and is not unique. It is given up to multiplication by a unit element in $\mathbb{Z}[t, t^{-1}]$. This would make adding the Alexander polynomials of two different links (and hence the skein relation in the next section) meaningless. The Conway normalized skein relation removes this ambiguity. One has $z = t^{1/2} - t^{-1/2}$.

- The signature of the Seifert matrix is another knot invariant (called knot signature).

2.5 Conway skein relation

One can compute the Alexander polynomial using skein relations.

The skein relation was first proved by Alexander himself but his result left unnoticed for decades when it was rediscovered by Conway in 1970.



Theorem 2.5.1 (Conway skein relation). *Let L_+, L_- and L_0 be link diagrams which differ only in a small neighborhood of a particular crossing, as in Figure 2.5. Then we have*

$$\nabla(L_+) - \nabla(L_-) = z\nabla(L_0) \tag{2.7}$$

$$\nabla(\bigcirc) = 1. \tag{2.8}$$

Moreover the above two equations uniquely determine ∇.

Note that if we start by declaring the skein relation, it is not clear if ∇ is well defined i.e. if we enumerate the crossings in different ways we would get the same result. (The proof is indeed complicated.)

Proof. We can use Seifert algorithm to construct Seifert surfaces $S_0, S_\pm$ for the three links which are identical outside a neighborhood of the crossing. Recall that for such a surface we have $2g+r-1 = 1-d+b$ and the left hand side is just the rank of the H_1 of the surface. So the rank of $H_1(S_\pm)$ is one more than that of $H_1(S_0)$. Let $f_1, \ldots, f_n$ be a basis for $H_1(S_0)$ and $e_\pm$ be the extra generator for $S_\pm$. Close examination shows that e_+^- goes under e_+ at the intersection while e_-^- goes over. Therefore $lk(e_+, e_+^-) = lk(e_-, e_-^-) - 1$. The rest of the Seifert matrices for L_+ and L_- are the same so a simple computation yields the skein relation. $\nabla_{L_+} - \nabla_{L_-} = (-t+t^{-1})(N-1)\nabla_{L_0} - (-t+t^{-1})N\nabla_{L_0} = (t-t^{-1})\nabla_{L_0} = z\nabla_{L_0}$.

Why uniquely determined? □

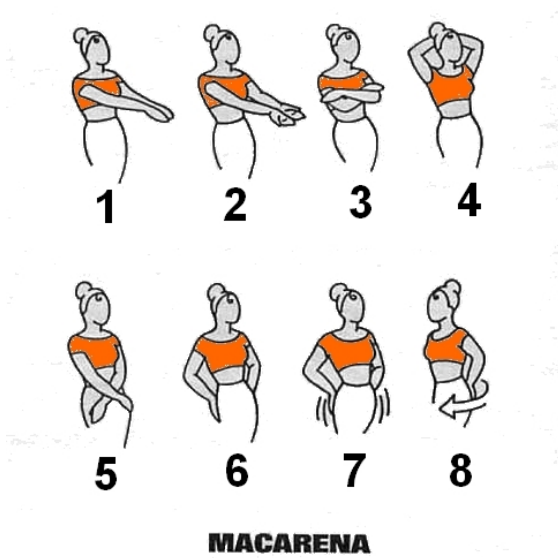


Figure 2.1: Skein relation (The joke is due to Ron Fintushel.)

Example 2.5.2. *Let's compute the Alexander polynomials of the Hopf link and the trefoil using the skein relation. If we pick a positive crossing in the Hopf link then L_- is an unlink with two components and L_0 is isotopic to the unknot so $\nabla_{Hopf} = z$.*
If we pick a positive crossing on the trefoil K then $K_- \simeq \bigcirc$ while K_0 is the Hopf link. Therefore we have $\nabla_{trefoil} = z^2 + 1$. These values agree with the computations using the Seifert matrix.

Example 2.5.3. *If T_n denotes the twist knot with n half twists then we have $L_+ = T_n$, $L_- = T_{n-2}$ and L_0 is the Hopf link. So we get $\nabla_{T_n} = -z^2 + \nabla_{T_{n-2}}$. We also have $T_0 = \bigcirc$ and T_1 is the trefoil.*

Exercise 2.5.4. *Compute the Alexander polynomial of the figure eight knot and of the Whitehead link. Show that the Alexander polynomial of the Kinoshita-Terasaka knot in picture 2.5 equals 1. Why is not isotopic to unknot.*

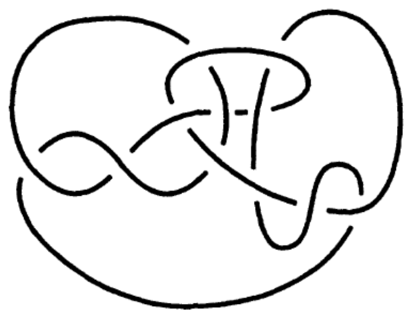


Figure 2.2: Kinoshita-Terasaka knot has the same Alexander polynomial as the unknot but has genus 2.

Proposition 2.5.5. *For a knot K we have $\nabla_K(1) = 1$.*

Exercise 2.5.6. *Prove Prop. 2.4.10 using the skein relation.*

2.6 Knot and link groups

By the complement $X(L)$ of a link L we mean S^3 minus a tubular neighborhood of the link. It is a 3-manifold with boundary, it's boundary consists of k tori where k is the number of components of the link.

If two links are equivalent then their complements are homeomorphic. However the reverse does not hold in general (for knots that are not prime). If $\phi : X(K) \to X(K')$ is a homeomorphism then it induces a homeomorphism $\partial X(K) \to \partial X(K')$ between the torus boundaries T, T'. This latter map can be extended to solid B, B' tori bounding T, T'. This way one obtains a map $\tilde{\phi} : X(K) \cup B \to X(K') \cup B'$.

Definition 2.6.1. *The group of a link L is $\pi(K) := \pi_1(X(L))$.*

It is of course a knot invariant since an equivalence between two knots induces an equivalence between their knot groups. It is a very powerful invariant.

Definition 2.6.2. *For a knot $K \subset S^3$, a meridian is an oriented simple closed curve lying on the boundary T of a tubular neighborhood of the knot which doesn't*

separate T into two parts and bounds a disk embedded in the tubular neighborhood . It is oriented so that its linking number with the knot is $+1$.
A longitude λ for K is another oriented circle in T such that there is an embedded annulus A in the tubular neighborhood such that $\partial A = K \cup -\lambda$ and such that $lk(K, \lambda) = 0$.

Exercise 2.6.3. *Show that the first homology of any knot complement is isomorphic to $\mathbb{Z}$ and is generated by a meridian of the knot.*

Because H_1 is the abelianization of π_1 it follows that all knot groups are infinite. This also means that the longitude of the knot is homologous to zero, which also follows from the existence of Sifert surfaces. However the longitude is not *homotopic* to zero unless K is the unknot.

We have $\pi(\bigcirc) \cong \mathbb{Z}$ because $X(\bigcirc)$ is isotopic to a solid torus (with the "solid" part lying "outside" the tubular neighborhood of the knot).

Proposition 2.6.4. *A knot K whose group is isomorphic to $\mathbb{Z}$ is equivalent to the unknot.*

What about a link? To prove this we need the following theorem.

Theorem 2.6.5 (Papakyriakopoulos)**.** *If M is a 3-manifold with boundary such that the induced map $\iota_* : \pi_1(\partial M) \to \pi_1(M)$ is not injective then there is an* embedded *disk in M whose boundary lies on ∂M and such that $[\partial D] \in \pi_1(\partial M)$ is nontrivial.*

Now to prove the proposition 2.6.4, let $M = X(K)$. The induced map $\pi_1(\partial M) = \pi_1(T^2) = \mathbb{Z}^2 \to \pi_1(M) = \pi(\bigcirc) = \mathbb{Z}$ cannot be injective so by the loop theorem there is an embedded disk D in M whose boundary is not homotopic to zero in ∂M. This can not be a meridian for K so (because the meridian is nontrivial in $\pi_1(X(K))$), it is homotopic to a longitude λ for K. Adding the annulus comming from the definition of a longitude to D we get an embedded disk in S^3 which bounds K.

Proposition 2.6.6. *A knot complement has trivial higher homotopy groups.*

Proof. For π_2 this follows from Sphere theorem of Papakyriakopoulos which asserts that if a 3-manifold has nontrivial π_2 then it contains an embedded homotopically nontrivial sphere. For the higher groups one can look at the universal cover of the complement and use Hurewicz's theorem which says for a simply connected cell complex the first nonvanishing homology and homotopy groups occur in the same dimension and are isomorphic. □

Two non-isotopic knots that are not prime may have homeomorphic complements. To observe this first note that $X(K)$ and $X(K^*)$ are homeomorphic (by an orientation reversing homeomorphism) and $X(-K)$ is the same as $X(K)$.

Also $X(K\#K')$ can be obtained by cutting the boundaries of $X(K)$ and $X(K')$ along a meridian and then gluing them to each other. (Boundary connected sum)

Now take any two knots K, K' and consider $K\#K'$ and $K\# - K'^*$. The complements of these two knots are obtained by taking boundary connected sums of $X(K), X(K')$ resp. $X(K), -X(K')$. The results are homotopy equivalent spaces. This is because the orientation reversing map on the anulus is *homotopic* to the identity. However the two composite knots are in general not isotopic, for example when both K, K' are the trefoil, as can be shown by the Jones polynomial.

Proposition 2.6.7. *Two prime knots are equivalent iff they have isomorphic groups.*

This theorem follows from the following two.

Theorem 2.6.8 (Whitten)**.** *If K, K' are two prime knots whose groups are isomorphic to each other then their complements are homeomorphic.*

Theorem 2.6.9 (Gordon, Luecke)**.** *If two knots K, K' are prime and ϕ : $X(K) \to X(K')$ is a homeomorphism then ϕ can be extended to a self homeomorphism of S^3 which sends K to K'.*

It is possible that two different links have homeomorphic complements, as in figure 2.6.

Figure 2.3: These two different links have homeomorphic complements.

Wirtinger presentation for link groups

We now describe a method due to Wirtinger to find generators and relations for a knot group, given a diagram of the knot. We then dicuss the problem of deciding when two group presentations yield isomorphic groups.

Start with a diagram D for your link L. At each crossing assign a generator to the over crossing arc and two generators to the two sections of the under crossing one. Each such generator corresponds to a loop that starts from the point at infinity in S^3 (thought of as lying above the plane of knot diagram), goes around the given segment of L and moves back to the point at infinity.

It is easy to see that these generators satisfy the relations as in Figure 2.6.

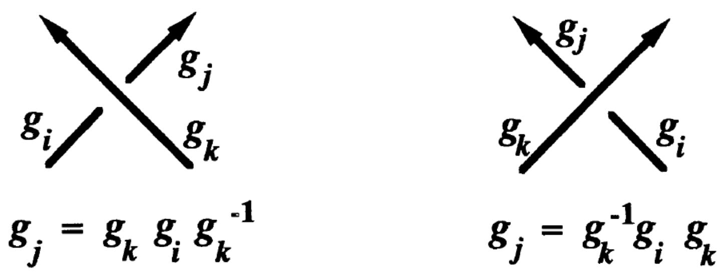


Figure 2.4: Wirtinger generators and relations

Remember that Van Kampen's theorem shows that if X is a topological space and $X = A \cup B$ where both A, B contain the basepoint and $A, B, A \cap B$ are connected such that the

$$\pi_1(X) \simeq \frac{\pi_1(A) * \pi_1(B)}{\{\iota_A(\alpha) = \iota_B(\alpha) | \alpha \in \pi_1(A \cap B)\}} \tag{2.9}$$

where ι_A, ι_B are maps from $\pi_1(A \cap B)$ to $\pi_1(A)$ and $\pi_1(B)$ respectively.

Proposition 2.6.10. *Wirtinger relations give all the relations in $\pi(K)$ between the generators g_i.*

Proof. We use Van Kampen's theorem. Consider vertical solid cylinders $S_1, \ldots S_n$ encircling each and every crossing in the diagram. Let B be a small neighborhood of $\cup S_i$. Each intersection $B \cap S_i$ is connected. Choose the basepoint to be at infinity and so contained in each of the S_i as well as in B. $\pi_1(S_i \backslash L)$ is given by Wirtinger presentation (as argued above) and using Van Kampen's for S_i's and B theorem shows that the link group is indeed given by the Wirtinger presentation. □

Example 2.6.11.
- *For the trefoil we have three generators g_1, g_2, g_3 and the generators are $g_3 g_1 g_3^{-1} g_2^{-1}$ and $g_1 g_2 g_1^{-1} g_3^{-1}$. We can define a homomorphism into S_3 by $g_1 \to (1, 2)$, $g_2 \to (2, 3)$, $g_3 \to (3, 1)$.*
- *For the Hopf link*

It follows that the group of the unlinked union of two links is the free product of the two links groups. Also the group of $K \# K'$ is the quotient of $\pi(K) * \pi(K')$ given by identifying the generators assigned to the edges that are connected to each other in the connected sum.

It follows that all the generators for a given component of the link lie in the same conjugacy class. In particular a knot group has only one conjugacy class.

In general it is not easy to decide if two group presentations yield isotopic groups (Tietze relations). However for a knot group one can consider homomorphisms into the symmetric group.

2.7 Fox calculus and Alexander polynomial

In this section we see that one can obtain the Alexander polynomial from a knot group using a formal calculus due to Fox. (It is also called free calculus.)

We want to define partial derivatives of words in letters x_i. We define $\frac{\partial x_i}{\partial x_j} = \delta_{i,j}$, $\frac{\partial x_i^{-1}}{\partial x_j} = -\delta_{i,j} x_i^{-2}$ and $\frac{\partial uv}{\partial x_i} = u\partial_{x_i} v + \partial_{x_i} u.v$.

Let $\pi(K)$ be given by n generators $x_1, \ldots, x_n$ and m relations $r_1, \ldots, r_m$. We can make an $n \times m$ matrix $J = (\partial_{x_i} r_j)_{i,j}$. Then we abelianize J into J' by sending $x_i \to t$. Now each $(n-1) \times (n-1)$ minor of J' gives the Alexander poly.

As an example we compute the Alexander polynomials of torus knots $T_{p,q}$.

2.8 Temperley-Lieb algebra and the Jones polynomial

The Temperly-Lieb algebra TL_n consists of $\mathbb{C}[q, q^{-1}]$-linear sums of flat (n, n)-tangles, so it's an algebra over $\mathbb{C}[q, q^{-1}]$. (A flat tangle is a tangle which admits a diagram without any crossings.) The product is given by composing the tangles. One declares that an element a with k embedded circles in it equals τ times a with the circles removed, where $\tau \in TL_n$ is a parameter. We take $\tau = -q^{-2} - q^2$. TL_n is generated by elements $U_1, \ldots, U_{n-1}$ where $U_i = \cup_i \cap_i$. The relations among these generators are as follows and are easy to check.

- $U_i^2 = \tau U_i$
- $U_i U_{i\pm1} U_i = U_i$
- $U_i U_j = U_j U_i$ where $|i - j| > 1$.

The only element of TL_n which has an inverse is the identity. Number of flat (n, n)-tangles is the Catalan number C_n. This is because we can move the top part of a flat tangle to sit next to the bottom part and this way we get a crossingless matching.

The closure of a flat tangle x is an unlink with some k components. We define the trace $trx = \tau^{k-1}$. We linearly extend over the whole TL_n.

Proposition 2.8.1. *We have a homomorphism $\phi : Br_n \to TL_n$ given by $\phi(\sigma_i) = qU_i + q^{-1}1$ and $\phi(\sigma_i^{-1}) = q^{-1}U_i + q1$. (Homomorphism in this case means that $\phi(xy) = \phi(x)\phi(y)$.)*

Proof. We verify the relations in the braid group. We have
$\phi(\sigma_i)\phi(\sigma_i^{-1}) = U_iU_i + 1 + (q^2 + q^{-2})U_i = 1$. Now
$\phi(\sigma_i\sigma_{i+1}\sigma_i) = (qU_i+q^{-1}1)(qU_{i+1}+q^{-1}1)(qU_i+q^{-1}1) = q^3U_iU_{i+1}U_i+qU_iU_{i+1}+ q^{-1}U_i + q^{-3}1 + qU_{i+1}U_i + q^{-1}U_{i+1} + qU_i^2 + q^{-1}U_i$.
The last two terms are equal to $-q^3U_i$ and the first term equals q^3U_i so they cancel each other and we are left with

$$qU_iU_{i+1} + q^{-1}U_i + q^{-3}1 + qU_{i+1}U_i + q^{-1}U_{i+1}.$$

We see that this is symmetric in exchanging i and $i + 1$ and so the result follows. □

Definition 2.8.2. *The writhe $w(D)$ of an oriented link diagram D is the number of positive crossings minus the number f negative crossings in D.*

Note that the writhe of a diagram is not changed under RII and RIII moves but it is indeed changed under RI. So it is not a link invariant.

Definition 2.8.3. *If a link L is given by the closure of a braid β then Kauffman bracket $< L >$ of L if we set*

$$< L >= tr\, \phi(\beta) \tag{2.10}$$

The Jones polynomial of L is defined to be $J(L)(t) = -q^{-3w(L)} < L >$ where $t = q^{-4}$.

Note that the Kauffman bracket doesn't care about the orientation of the link while the Jones polynomial does (because it involves the writhe).

Proposition 2.8.4. *The Kauffman bracket is invariant under Markov move I (conjugation). The Jones polynomial is a link invariant with values in $\mathbb{Z}[t^{-1/2}, t^{1/2}]$. Moreover the Jones polynomial of a link with an odd number of components (such as a knot) lies in $\mathbb{Z}[t^{-1}, t]$.*

Proof. If σ is another braid then $tr\phi(\sigma\beta\sigma^{-1}) = tr\, \phi(\sigma)\phi(\beta)\phi(\sigma^{-1}) = tr\phi(\sigma^{-1})\phi(\sigma)\phi(\beta) = tr\phi(\beta)$. The second to last equality is because trace is given by closing the flat tangle and when we do this to $\phi(\sigma^{-1})\phi(\beta)\phi(\sigma)$, the first and last terms come together. (In other words trace is invariant under cyclic permutation.) This establishes the first statement.

Since $<>$ is invariant under MI and writhe is not changed under this move, to prove the invariance of J we only need to show that the Jones polynomial is invariant uner MII. Let $\beta' \in Br_{m+1}$ be obtained from $\beta \in Br_m$ by MII. Then $\phi(\beta') = q\bar{\phi}(\beta)U_n + q^{-1}\bar{\phi}(\beta)$ where the bar denotes the map $TL_n \to TL_{n+1}$ given by adding a strand to the right. The closure of the $\bar{\phi}(\beta)U_n$ is the same as the closure of $\phi(\beta)$ while the closure of $\bar{\phi}(\beta)$ is the closure of $\phi(\beta)$ union a circle. So $tr\phi(\beta') = qtr\phi(\beta) + q^{-1}\tau tr\phi(\beta) = (q - q^{-3} - q)tr\phi(\beta)$. We see that if a link L' is obtained from L by RIII then

$$< L' >= q^{-3} < L > . \tag{2.11}$$

Since β' has one more positive crossing than β does, its writhe is one more than that of β and so the extra q^{-3} factor is canceled by $^{-w(L)}$ factor in the definition of the Jones polynomial. □

Example 2.8.5. *Computation for trefoil and Hopf.*

Let $L = L_+$ be a link diagram and consider a specific positive crossing in D. Let L_0 be the result of replacing the crossing with two vertical lines and L_1 the result of replacing it with a cup-cap pair as in Figure 2.8. Note that if you rotate this picture 90 degrees then the 0 and 1 resolutions are exchanged.

Proposition 2.8.6 (Skein relation for Kauffman bracket)**.** *The Kauffman bracket satisfies the following relations and is uniquely determined by them.*

$$< L_+ >= q < L_0 > +q^{-1} < L_1 > \tag{2.12}$$

and $< \bigcirc >= 1$.

Proof. If the diagram is in braid position then this follows immediately from the definition of the homomorphism ϕ. □

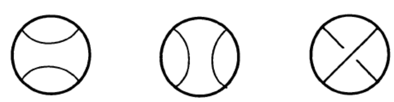


Figure 2.5: From right to left: L_+, L_0, L_1

Proposition 2.8.7 (Skein relations for the Jones polynomial)**.**

$$J_{L_+} = t^{1/2} J_{L_0} + t J_{L_1} \tag{2.13}$$

$$J_{L_-} = t^{-1} J_{L_1} + t^{-1/2} J_{L_0} \tag{2.14}$$

$$t^{-1} J_{L_+} - t J_{L_-} = (t^{-1/2} - t^{1/2}) J_{L_0} \tag{2.15}$$

Proof. For the first equation use the definition of the Jones polynomial and the Kauffman skein relation (compute in terms of q and then substitute with t). Proof of the second one is similar with the difference that the 0-resolution for L_+ is the same as the the 1-resolution for L_-.

For the last equation, expand and simplify $q^4 < L_+ > -q^{-4} < L_- >$ using the Kauffman relation. Note that the writhe of L_+ is 2 more than that of □

Exercise 2.8.8. *Show that $J_L(1) = 2^l$ where L is the number of components of L.*

2.9 State sum model for the Jones polynomial

Imagine we have a link diagram D with n crossings. We can apply the Kauffman skein relation to each and every crossing in D. Each crossing has two different resolutions so we end up with 2^n "states" each one of which is a "full resolution" of D. In a full resolution every crossing in D is resolved and so it is a disjoint union of embedded circles in the plane. We can dnote each state by an element $I \in \{0,1\}^n$. Each such I contributes a term to the Kauffman bracket. The coefficient for the resolution I from the Kauffman bracket is $q^{1(I)-0(I)}$ and we are left with some $s(I)$ circles in the plane. So the term is $q^{0(I)-1(I)}(-q^2-q^{-2})^{s(I)-1}$. From this we get

$$< L >= \sum_{I\in\{0,1\}^n} q^{0(I)-1(I)}(-q^2 - q^{-2})^{s(I)-1} \tag{2.16}$$

2.10 The Jones polynomial and the quantum $\mathfrak{sl}_2$

If V is the fundamental representation of quantum $\mathfrak{sl}_2$ then

$$Inv(V^{2n}) \subset V^{2n}(0)$$

.

$Inv(V^{2n}) \cong K(H^m - mod)$ (finitely generated modules) More precisely The Grothemdieck group is generated as a $\mathbb{Z}[q, q^{-1}]$-module by $[\mathbb{Z}(a)]$ for m-crossingless matchings a. What is the space of invariants for V^{2m+1}?

$V^n(n-2k) \cong K(\varnothing^{n-k,k}) \otimes_{\mathbb{Z}} \mathbb{C}$. $\varnothing^{n,n-k} \cong A_{n-k,k} - mod$ finite dimensional modules. $A_{n-k,k}$ is the Braiden algebra. $\varnothing = \varnothing(\mathfrak{g})$ consists of finitely generated $\mathfrak{g}$-modules M for which

- The action of $\mathfrak{h}$ on M is diagonalizable.
- $\dim U(\mathfrak{n}^+)v < \infty$ for all $v \in M$.

$H^n \otimes_{\mathbb{Z}} \mathbb{C} \cong eA_{n,n}e$ for some idempotent e. Stroppel's $\mathcal{K}^n \cong A_{n,n}$.

Chen-Khovanov define, in a down to earth way, graded rings $A^{n-k,k}$. $K(A^{n-k,k} - mod) \otimes \mathbb{C} \cong V^n$. $A^n = \prod_k A^{n-k,k}$. They assign to an (m,n)-tangle a complex of graded (A^m, A^n)-bimodules. Stroppel's $\mathcal{K}^n \cong A^{n,n} \otimes \mathbb{C}$.

2.11 Skein modules

Let's consider the $\mathbb{Z}[z, v, z^{-1}, v^{-1}]$ module S generated by the equivalence classes of oriented links modulo the relation

$$vL_+ + v^{-1}L_- = zL_0.$$

Theorem 2.11.1. *S is generated is a free module with a basis given by the unknot. Therefore the equivalence class of each link type L is a polynomial in z, v, z^{-1}, v^{-1} times the unknot and this polynomial is an invariant of the link which is the same as.*

the statement and proof for tangles

what is the inv for tangles and how is it related to other tangle poly's?

2.12 [UNFINISHED] Jones invariant of tangles by capping

In this section we want to extent the Jones polynomial to tangles in the most conceptually simple manner. To this end we cap the tangle with crossingless matchings.

Whenever we have an invariant Ψ of links and we want to generalize it to an invariant $\tilde{\Psi}$ of tangles, $\tilde{\Psi}$ must satisfy two conditions

i. $\tilde{\Psi}(L) = \Psi(L)$ for any link L.

ii. Functoriality: $\tilde{\Psi}(T_2 \circ T_1) = \tilde{\Psi}(T_2) \circ \tilde{\Psi}(T_1)$.

Definition 2.12.1. *A crossingless matching with n endpoints is a $(n, 0)$-tangle which doesn't have any crossings.*

The number of isotopy classes of crossingless matchings with n endpoints equals the Catalan number C_n.

Let T be a general tangle. For the simplicity of presentation and because RNA molecules are represented by tangles with no outgoing points, we restrict our attention to the case that T is an $(n, 0)$ tangle.

What the Jones polynomial assigns to a link, which is a $(0, 0)$-tangle, is an element of the base field of the Temperley-Lieb algebra. This base field can be regarded as TL_0. Similarly what we assign to a $(n, 0)$ tangle will be a map $TL_n \to TL_0$.

Note that any crossingless matching D with n endpoints gives a map $\phi_D : TL_n \to TL_0$ which is given by capping a basis element X of TL_n (which is a flat (n, n)-tangle) on both sides by D and counting the number of resulting circles. More precisely

$$\phi_D(X) = \tau^{(\# D^t X D)-1} \tag{2.17}$$

where $\#$ denotes the number of components.

Definition 2.12.2. *The Jones invariant of a $(n, 0)$-tangles T is an element of* $\mathrm{Hom}(TL_n, TL_0)$ *given by*

$$\sum_D J(D^t T) \cdot \phi_D \tag{2.18}$$

where the sum is over all the C_n isotopy classes of crossingless matchings. It is evident that if $n = 0$ and so T is a link then the above definition gives the original Jones polynomial.

We next consider functoriality. Let E be a $(0, n)$-tangle a definition similar to the above gives an invariant $J(E) : TL_0 \to TL_n$. We have to check that $J(D) \circ J(E)(1)$ is the Jones polynomial of the link DE. This is equivalent to

Proposition 2.12.3. *The automorphism $\sum_{D,E} \phi_E \circ \phi_D$ of TL_n equals the identity map.*

2.13 The Jones and Alexander polynomials from R-matrices

Let V be a two dimensional vector space over $\mathbb{C}$ with a basis e_1, e_2. We can define a representation of the symmetric group S_n on $V^{\otimes n}$ which sends the transposition $s_i \in S_n$ to the linear map that exchanges the ith and $i+1$th factors in $V^{\otimes n}$. More precisely let $P : V \otimes V \to V \otimes V$ be the map $P(\sum_i x_i \otimes y_i) = \sum y_i \otimes x_i$ and define a homomorphism $\phi : S_n \to \mathrm{End}(V^{\otimes n})$ by

$$\phi(s_i) = id^{\otimes i-1} \otimes P \otimes id^{\otimes n-i-1}. \tag{2.19}$$

We can compose ϕ with the homomorphism $Br_n \to S_n$ to get a representation of the braid group. But we want a representation which doesn't necessarily factor through a representation of the symmetric group. To this end we consider an arbitrary linear map $R : V \otimes V \to V \otimes V$ and ask ourselves for which R does the map

$$\psi(\sigma_i) = id^{\otimes i-1} \otimes R \otimes id^{\otimes n-i-1} \tag{2.20}$$

give a representation of the braid group Br_n on $V^{\otimes n}$.

It is easy to see that for any R such a ψ satisfies the relation $\psi(\sigma_i \sigma_j) = \psi(\sigma_j \sigma_i)$ for $|i - j| > 1$. For ψ to satisfy the braid relation, R has to satisfy the following equation

$$(R \otimes id)(id \otimes R)(R \otimes id) = (id \otimes R)(R \otimes id)(id \otimes R) \tag{2.21}$$

which is called the Yang-Baxter equation. A solution to this equation is called an R-matrix.

Definition 2.13.1. *We say that an R-matrix satisfies charge conservation if the $e_k \otimes e_l$ coefficient in the expansion of $R(e_i \otimes e_j)$ is nonzero then $i + j = k + l$. Such an R-matrix has the following form in the standard basis of $V \otimes V$.*

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix} \tag{2.22}$$

Proposition 2.13.2. *An R-matrix which satisfies charge preservation has one of the following two forms:*
$b = 0, e = a - cd/a, f = a$
$b = 0, e = a - cd/a, f = -cd/a$

We set $c = d = a^2 = t$. If a link is given by the closure of a braid $\beta \in Br_n$ we want to be able to get an invariant of L from $tr\psi(\beta)$. Markov move I is satisfied because trace is invariant under conjugation of matrices.

We instead consider

$$tr(h^n\psi(\beta)) \tag{2.23}$$

where $h = diag(t^{-1/2}, t^{1/2}) \in End(V)$.

Theorem 2.13.3. *The assignment $\beta \to tr(h^n\psi(\beta))$ is invariant under Markov moves so it gives a link invariant (for the closure of β). This invariant equals $t^{1/2} + t^{-1/2}$ times the Jones polynomial.*

Proof. Only equivalence to the Jones polynomial has not been proved. To this end we show that our invariant satisfies the Kaufman skein relation. □

2.14 Tangle invariants from R-matrices

If we define Jones by capping with crossingless matchings we get a linear combination of these crossingless matchings which can be regaded as a map from $TL_n \to \mathbb{C}[q, q^{-1}]$ or from V^{2n} to $\mathbb{C}[q, q^{-1}]$. How to prove functoriality?

Let V be a vector space over $\mathbb{C}$. To an oriented (m, n)-tangle T. The lower end of T consists of m points each one of them having a sign s_i which is positive if the strand of T at the point is pointing downwards and negative otherwise. We have a similar vector of signs $s'_1, \dots, s'_n$ fr the upper end. We set V^{s_i} to be V if s_i is positive and V^* otherwise.

We want to assign a linear map

$$\psi(T) = V^{s_1} \otimes \cdots \otimes V^{s_m} \to V^{s'_1} \otimes \cdots \otimes V^{s'_n}. \tag{2.24}$$

We will do this by decomposing T into a composition of elementary tangle and assigning linear maps to each elementary tangle.

Let $\cap_i$ denote a cap tangle with counterlocwise orientation and $-\cap_i$ the same tangle with clockwise orientation. Similarly for $\cup_i$ and $-\cup_i$.

We start from arbitrary linear maps $R \in \mathrm{End}(V \otimes V), h \in \mathrm{End}(V)$. Let the maps $u : \mathbb{C} \to V^* \otimes V$, $u' : \mathbb{C} \to V \otimes V^*$, $\epsilon : V \otimes V^* \to \mathbb{C}$, $\epsilon' : V^* \otimes V \to \mathbb{C}$ be given by:
$u(1) = \sum e_i^* \otimes h(e_i)$,
$u'(1) = \sum e_i \otimes e_i^*$,
$\epsilon(v, f) = f(h(v))$,
$\epsilon'(f, v) = f(v)$.

We set
$\psi(\sigma_i) = id^{\otimes i-1} \otimes R \otimes id^{\otimes n-i-1}$.
$\psi(\sigma_i^{-1}) = id^{\otimes i-1} \otimes R^{-1} \otimes id^{\otimes n-i-1}$.
$\psi(\cup_i) = id^{\otimes i-1} \otimes u' \otimes id^{\otimes n-i-1}$
$\psi(-\cup_i) = id^{\otimes i-1} \otimes u \otimes id^{\otimes n-i-1}$
$\psi(\cap_i) = id^{\otimes i-1} \otimes \epsilon \otimes id^{\otimes n-i-1}$
$\psi(-\cap_i) = id^{\otimes i-1} \otimes \epsilon' \otimes id^{\otimes n-i-1}$

For a linear map $A \in \mathrm{End}(V_1 \otimes V_2) = V_1^* \otimes V_1 \otimes V_2^* \otimes V_2$ let
$A_1 \in \mathrm{Hom}(V_1^* \otimes V_1, V_2 \otimes V_2^*)$
$A_2 \in \mathrm{Hom}(V_2 \otimes V_2^*, V_1^* \otimes V_1)$

Theorem 2.14.1 (Turaev)**.** *If R satisfies the Yang-Baxter equation and R, h satisfy the following equations*

$$R \circ (h \otimes h) = (h \otimes h) \circ R \tag{2.25}$$

$$tr_2(id_V \otimes h) \circ R^{\pm 1} = id_V \tag{2.26}$$

$$R_1^{-1} \circ ((id_V \otimes h) \circ R \circ (h^{-1}) \otimes id_V)_2 = id_v \otimes id_V. \tag{2.27}$$

then ψ is a tangle invariant. Moreover the R, h from the last section satisfy these equation and the invariant they give to a $(0,0)$-tangle is its Jones polynomial.

For an $(n,0)$-tangle T such as one associated to an RNA molecule we get a linear map $V^{\otimes n} \to \mathbb{C}$. We can write this map as inner product with a unique element v of $V^{\otimes n}$ so we can regard v as the invariant associated to T.

2.15 From links to graphs and back

To any link diagram one can associate a weighted edge graph embedded in the plane called the Tait graph and vice versa.

Starting from a link diagram, we first need a checkerboard coloring of the diagram. This means that we color the faces (connected components of the complement) of the link projection with black and white in such a way that adjacent components have different colors.

To obtain the Tait graph we associate a vertex to each black component and an edge to each crossing. The edges are marked with $+$ or $-$ depending whether the crossing is right or left handed. If we use the white regions instead we get the dual graph.

Associating links to signed graphs

To any planar signed graph G we can associate a link diagram as follows. We first take the medial graph G_m of G. G_m has a 4-valent vertex in the middle of each edge of G. The edges are the drawn by following the face boundaries of G.

Proposition 2.15.1.

- $D(T(D)) = D$
- $T(D(G)) \in \{G, G^*\}$
- $D(G) = D(G^*)$

Associating fatgraphs to link diagrams

Let D be a link diagram with c crossings and let $Iin\{0,1\}^n$. As we know each such I gives us a complete resolution of the diagram into a set of disjoint closed curves in the plane. We can associate a fatgraph $G(D,I)$ to such a resolution as follows. To each connected component of the complement of the set of circles we assign a vertex. In place of each crossing we put an edge between the corresponding vertices. The vertices emanating from any vertex have a natural cyclic order that comes from the orientation of the plane.

There are 2^c different such fatgraphs. We note that both a Seifert surface and the Tait graph of D are among these fatgraphs. Namely if D is a diagram of an oriented link then we can choose I to be so that each crossing is resolved in a way which is compatible with the orientation. The surface associated $G(D,I)$ is the same as the one obtained from the Seifert algorithm.

2.16 Homfly polynomial

Homfly is a two variable invariant which includes both Alex and Jones. It is the most general poly inv which satisfies the skein relation.

Theorem 2.16.1. *There is a unique invariant of oriented links P taking values in the ring of Laurent polynomials in three variables such that $P(\bigcirc)=1$ and P satisfies the skein relation*

$$xP(L_+)+yP(L_-)+zP(L_0)=0. \tag{2.28}$$

2.17 Tutte polynomial and HOMFLY polynomial

2.18 Beyond the skein relation: Finite type invariants

Chapter 3

Properties of knots and links

3.1 Alternating knots

3.2 Fibred knots

3.3 Slice knots

Chapter 4

Surfaces

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