Chapter 3: Introduction to Classification

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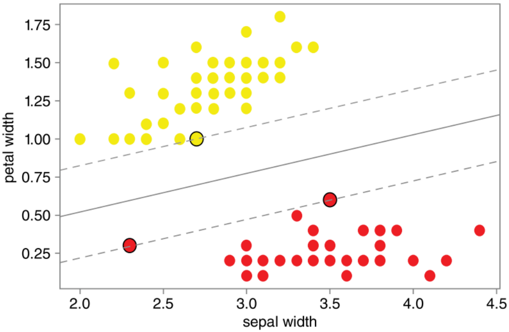

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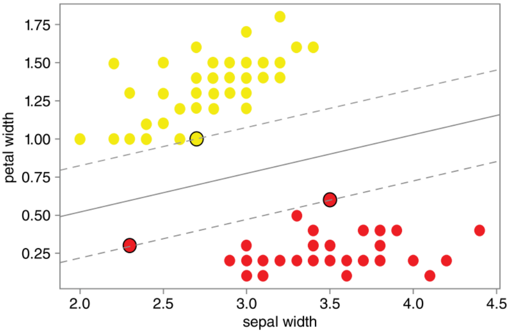

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- The points in either class that are closest to this hyperplane are called *support vectors*.

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- The distance between a point $\mathbf{x}$ and this hyperplane is given by $\frac{|\langle \mathbf{x}, \mathbf{a} \rangle + a_0|}{||\mathbf{a}||}$.
Therefore the minimum distance from the points in either class to the hyperplane equals $||\mathbf{a}||^{-1}$.

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- Equivalently consider $L(\mathbf{x}, \Lambda) = f(\mathbf{x}) - \sum_i \lambda_i g_i(\mathbf{x})$ and set $\nabla L = 0$.

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- It follows that if $\lambda_i \neq 0$ then $y_i(\mathbf{a}_0^t \mathbf{x}_i - a_0) = 1$ and thus $s_i = 0$.
- Therefore, if we add or remove points to our dataset, the solution to SVM is not changed, as long as these new points are farther from the hyperplane than the support vectors and "on the correct side".

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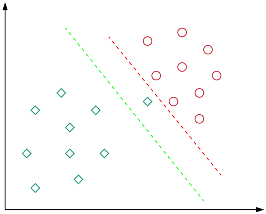

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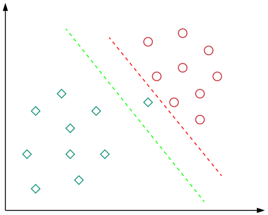

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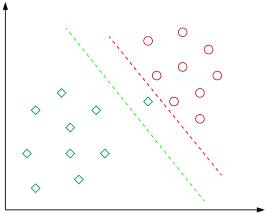

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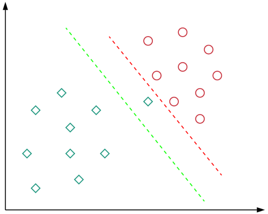

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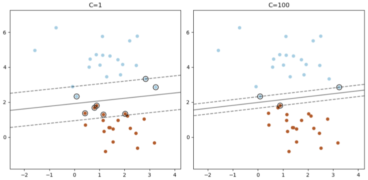

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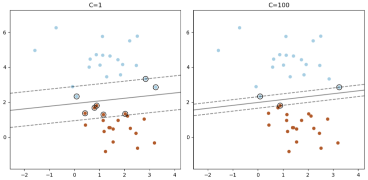

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- In Python, linear SVM is given by the class `sklearn.svm.LinearSVC`

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- Exercise: find the subgradient vector for the loss function (8).
- This method is available as `sklearn.linear_model.SGDClassifier(loss='hinge')`
- One advantage of this method is the ability to use SGD which scales well with the size of the training dataset.

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SVM for non-linearly separable classes

- Often the two classes cannot be separated by a hyperplane.
- In such cases we may hope that by adding more features, the two classes will become linearly separable.
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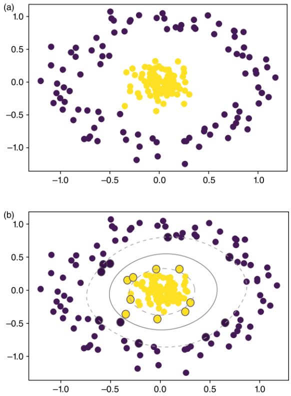
- Of course we have $\Phi(S) = P$.

Kernel functions in nonlinear SVM cont.

- An example of using the SVM with RBF kernel.



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In general SVM optimization problem is solved using Sequential Minimal Optimization method which reduces the problem into a sequence of optimization problems involving only two Lagrange multipliers at each step.
See: John C. Platt, Sequential Minimal Optimization: A Fast Algorithm for Training Support Vector Machines, 1998.
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- In the soft-margin SVR we want to minimize $\frac{1}{2}||\mathbf{a}||^2 + K\sum_i \max(|\mathbf{a}^t\mathbf{x}_i + a_0 - y_i| - \epsilon, 0)$

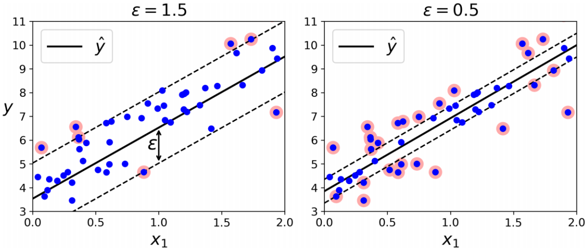

Figure: SVR for two different values of ϵ. Credit: A. Geron

- SVR can be used for nonlinear regression by replacing $\mathbf{a}^t\mathbf{x}$ with a kernel $K(\mathbf{a}, \mathbf{x})$!

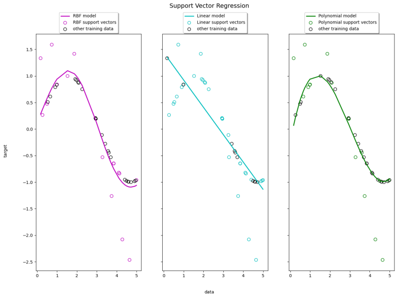

Figure: Kernel SVR for sinusoidal data (with added noise).

- Giving no penalty for when error is less than ϵ prevents unnecessary fluctuations of the predicted f.
- In Python (soft-margin) SVR is provided by the class `from sklearn.svm.SVR`

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- In this method the class that wins against the highest number of other classes, is assigned.

Adapting a binary classifier for multi-class classification, cont.

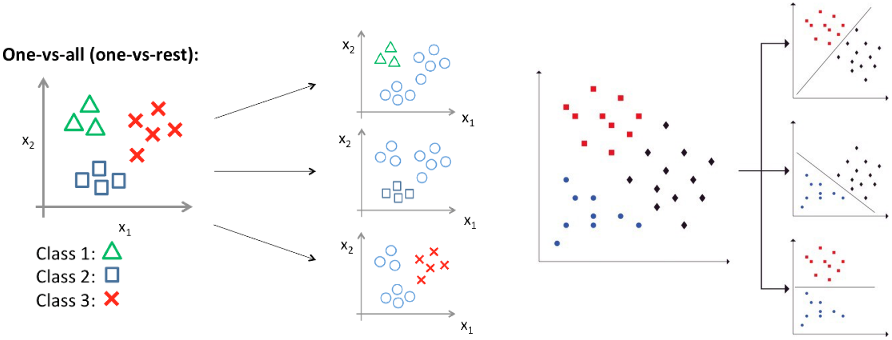

Figure: One-vs-All (left) and One-vs-One (right). Credit: Jatin Nanda, Zhongliang Zhang, *et al.*

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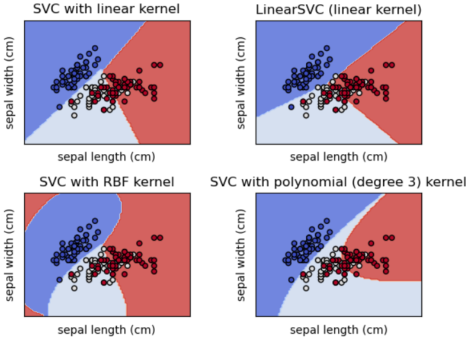

SVM quiz

Consider the following dataset:

feature1	feature2	class
2	0	good
0	1	bad
1	1	good

Compute the equation of the hard-margin SVM hyperplane for this dataset.

Solution to the quiz

We represent the two features by x_1, x_2. We want to find the hyperplane (a line in this case) of the form $a_1x_1 + a_2x_2 + a_0 = 0$ which maximizes $a_1^2 + a_2^2$, subject to the constraints of (2).
To simplify the solution note that we can assume that either $a_2 = 0$ or $a_2 = 1$, since we can divide by a_2 if it's not zero. Thus we have two cases $a_0 + a_1x_1 + x_2 = 0$ or $a_0 + a_1x_1 = 0$. We solve the problem for each situation separately and compare the results.
In the first case, we want to minimize $1 + a_1^2$ (equivalently $|a_1|$) subject to
$a_0 + 2a_1 + 0 \geq 1$,
$a_0 + 0 + 1 \leq -1$,
$a_0 + a_1 + 1 \geq 1$.
The region in $\mathbb{R}^2$ satisfying the above condition is bounded by the three lines given by when the above inequalities are qualities i.e
$a_0 + 2a_1 = 1$, $a_0 = -2$, $a_0 + a_1 = 0$. Drawing these lines we realize that the point subject to the above inequalities at which $\min |a_1|$ is achieved is $(-2, 2)$. This gives $\min ||\mathbf{a}||^2 = 1 + 2^2 = 5$, for the first case.

Solution to the quiz

In the second case, $||\mathbf{a}|| = a_1^2$.
The constraints in this case are:
$a_0 + 2a_1 \geq 1$,
$a_0 + 0 \leq -1$,
$a_0 + a_1 \geq 1$.
From the 2nd and 3rd inequalities we get $1 \leq a_0 + a_1 \leq a_1 - 1$ which implies $a_1 \geq 2$. Thus the smallest value of a_1 is 2, for which $a_0 = -1$. (This is achieved because (-1,2) satisfies the above inequalities. You can also draw the region in this case.) At this point $||\mathbf{a}||^2 = 2^2 = 4$. This is smaller than the first case and thus the solution is given by $-1 + 2x_1 = 0$.

Problems

1. Let $\{(\mathbf{x}_i, y_i)\}$ be a dataset such that $\mathbf{x}_i \in \mathbb{R}^d$ and $y_i \in \{-1, 1\}$. Set P, N consist of $\mathbf{x}_i$ for which y_i is positive, and negative respectively Suppose there is a linear function $f(\mathbf{x})$ such that for each i we have $sign(f(\mathbf{x}_i)) = sign(y_i)$. Then show that there is an SVM hyperplane for the dataset i.e. a hyperplane $H \subset \mathbb{R}^d$ such that $d(P, H) := \min_{\mathbf{x} \in H}\{d(\mathbf{x}, H)\} = \min_{\mathbf{y} \in N}\{d(\mathbf{y}, H)\} =: d(N, H)$ and moreover for any other hypersurface H' either $d(P, H') > d(P, H)$ or $d(N, H') > d(N, P)$.
2. With the same assumptions as above, let $\mathbf{x}_k, \mathbf{x}_l$ be such that $y_k y_l = -1$ and that $d(\mathbf{x}_k, \mathbf{x}_l) \leq d(\mathbf{x}_i, \mathbf{x}_j)$ for any $\mathbf{x}_i \in P, \mathbf{x}_j \in N$. Is it true that $\mathbf{x}_k, \mathbf{x}_l$ are support vectors for H?
3. Show that the loss function for soft-margin SVM is convex and then find subgradients for it, at point where it is not differentiable.
4. What is the soft-margin penalty for a point that lies on the boundary of the margin but on the wrong side?