Chapter 2. Regression

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- If X^tX is invertible then $\mathbf{a} = (X^tX)^{-1}X^t\mathbf{y}$. (The normal equation)

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- In case of collinearity, the entries of $(X^t X)^{-1}$ can be very large and will jump by small perturbations of data.

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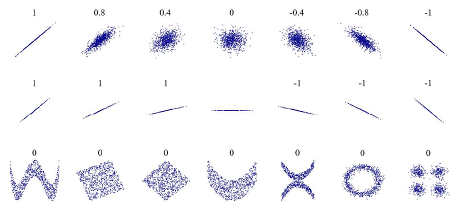

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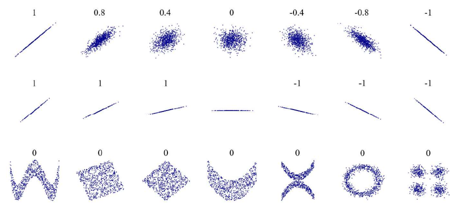

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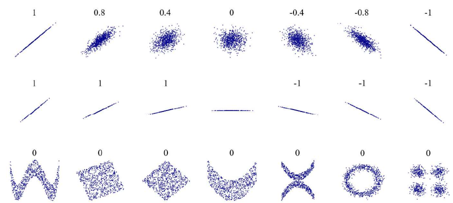

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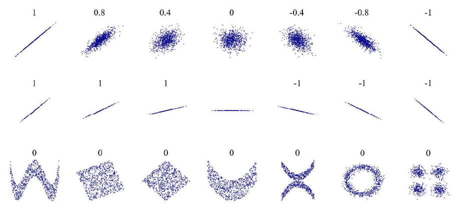

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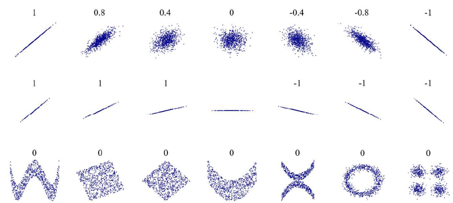

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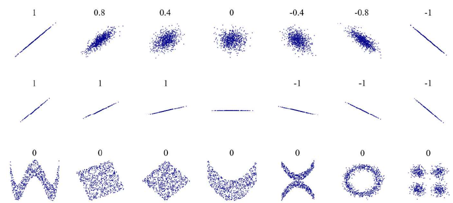

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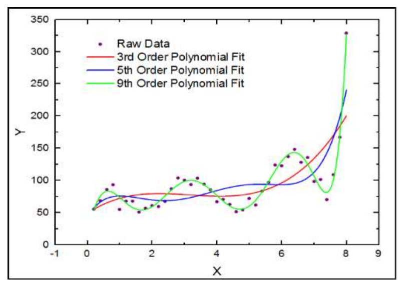

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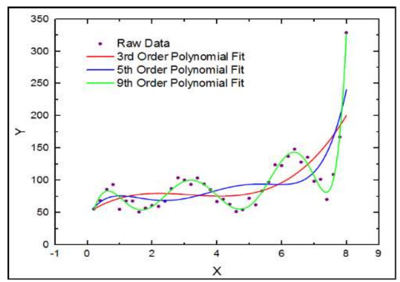

- In Python, polynomial features can be obtained using `sklearn.preprocessing.PolynomialFeatures`

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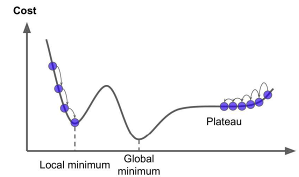

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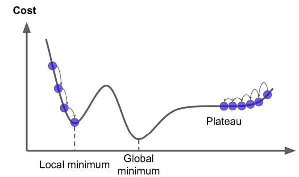

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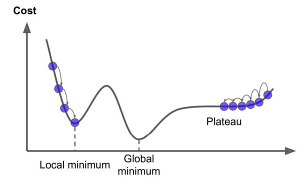

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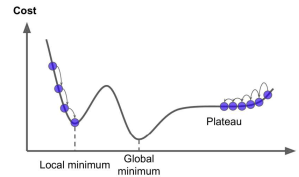

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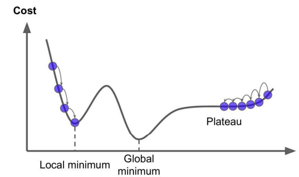

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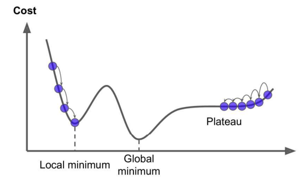

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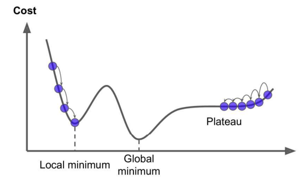

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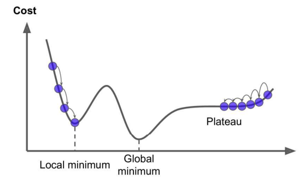

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- More complex (or rugged) fitness landscapes need more exploration.

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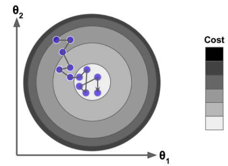

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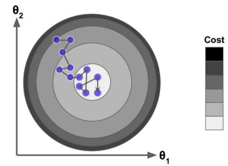

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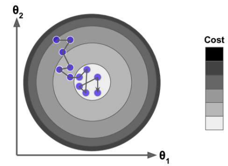

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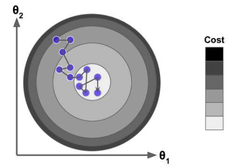

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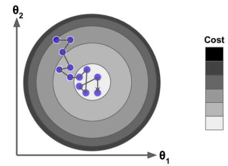

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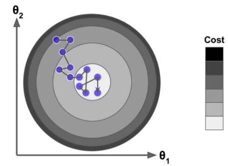

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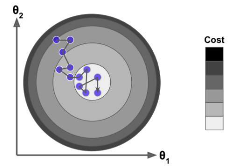

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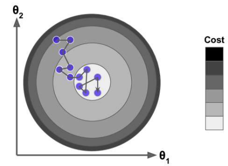

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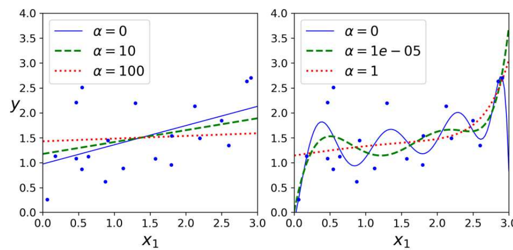

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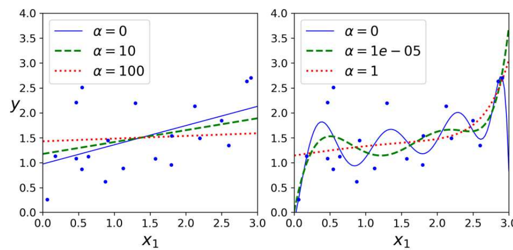

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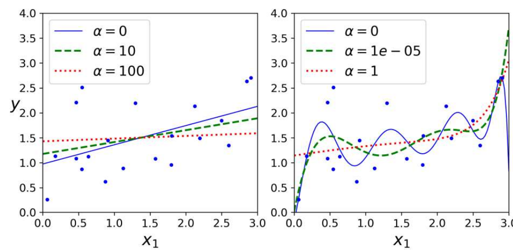

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Lasso Regression

- LASSO regression uses ℓ_1 norm of **a** instead of ℓ_2, for regularization:

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- Sparse models are more explainable because the relation between the features and the target variable is easier to see.

Gradient Descent for Lasso Regression

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- In Scikit-Learn, Lasso regression is provided by `SGDRegressor(penalty="l1")`

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- Elastic Net regularization uses a linear combination of Ridge and LASSO penalty terms:

$$\alpha \cdot ||\mathbf{a}||_{\ell_1} + \frac{1-r}{2}\alpha \cdot ||\mathbf{a}||_{\ell_2}^2 \tag{7}$$

where $0 \leq r \leq 1$